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Additively unique sets of prime numbers

Noah Lebowitz-Lockard

Department of Mathematics, University of Georgia
452 Boyd Graduate Studies Research Center
Athens, GA 30602, USA
noah.lebowitzl25@uga.edu

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Spiro proved that the identity function is the only multiplicative function with $f(p) \neq 0$ for some prime p and $f(p+q) = f(p) + f(q)$ for all prime p and q. We determine the sets S of primes for which restricting our condition to $f(p+q) = f(p) + f(q)$ for all $p, q \in S$ still implies that f is the identity function. We prove that S satisfies these conditions if and only if S contains every prime that is not the larger element of a twin prime pair and S contains 5 or 7.

Keywords: Additive uniqueness; prime numbers; multiplicative functions.

Mathematics Subject Classification 2010: 11B13

1. Introduction

Let S be a set of positive integers and $\mathcal{F}$ be a set of arithmetic functions. If the set S is large, then there should be very few functions $f \in \mathcal{F}$ satisfying $f(m+n) = f(m) + f(n)$ for all $m, n \in S$. This suggests the following definition.

Definition. If there is exactly one element $f \in \mathcal{F}$ satisfying $f(m+n) = f(m)+f(n)$ for all $m, n \in S$, then S is an *additive uniqueness set* for $\mathcal{F}$. (We use "AU" as a shorthand for "additively unique".)

Throughout the rest of this paper, we let $\mathcal{F}$ be the set of multiplicative functions that do not vanish on some prime. (Note that this is a stronger condition than a multiplicative function being nonvanishing.) No matter what set S is, the identity function $f \in \mathcal{F}$ satisfies $f(m+n) = f(m) + f(n)$. If S is additively unique, then the identity function is the only function satisfying this condition.

Spiro [7, Theorem 1] proved that the prime numbers are additively unique. The next logical question is which subsets of the set of prime numbers are additively

unique. Chen and Chen [1] proved the following result, which implies that odd primes are not AU as a consequence.

Theorem 1.1 ([1, Theorem 1]). *Let f be a multiplicative function for which $f(p) \neq 0$ for some odd prime p and*

$$f(p+q) = f(p) + f(q) \quad \text{for all odd primes } p, q.$$

Then, $f(n) = n$ for all n or

$$f(n) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Our result shows that almost all primes are "necessary" for AU in the sense that an additively unique set of primes must contain them.

Theorem 1.2. *A set S of primes is AU if and only if it contains every prime that is not the larger element of a twin prime pair and at least one element of $\{5, 7\}$.*

The proof of this result is very similar to Spiro's proof that the primes are AU and follow its structure closely.

2. Preliminary Results

To start the proof, we show that the primes listed in Theorem 1.2 are necessary. From here on, S will refer to a set of primes.

Lemma 2.1. *If S is AU, then S contains every prime that is not the larger element of a twin prime pair.*

Proof. Let p_0 be a prime that is not the larger element of a twin prime pair. Suppose $p_0 \notin S$. Consider the function

$$f(n) = \begin{cases} 1, & \text{if } n \in \{1, p_0\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that f is multiplicative. Let $p, q \in S$. Then, p and q are primes that are not equal to p_0. If $p_0 = 2$, then $p + q > p_0$. Otherwise, p_0 is odd. If $p + q = p_0$, then p or q must be 2. But, this is impossible because p_0 is not the larger element of a twin prime pair. Hence, $p+q \neq p_0$. So, $f(p+q) = f(p)+f(q) = 0$. Thus, $f \in \mathcal{F}$ even though f is not the identity function. If $p_0 \notin S$, then S is not AU. □

The next lemma would still hold if we replaced 5 and 7 with any twin prime pair $p, p+2$, but this would not give us any new information.

Lemma 2.2. *If S is AU, then S contains 5 or 7.*

Proof. Suppose $5, 7 \notin S$. Consider the function

$$f(n) = \begin{cases} 1, & \text{if } n \in \{1, 7\}, \\ 0, & \text{otherwise.} \end{cases}$$

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Once again, f is multiplicative. Let $p, q \in S$. Then, $p, q, p+q \neq 7$ because the only way to express 7 as the sum of two primes is $2+5$. So, $f \in \mathcal{F}$. Because f is not the identity function, S is not AU. □

Now we have the necessity of the primes in Theorem 1.2, we may spend the rest of the paper establishing their sufficiency. Over the next few lemmas, we show that if $f(m+n) = f(m) + f(n)$ for all m, n in an additively unique set S, then $f(n) = n$ for all $n \leq 23$.

Lemma 2.3. *If $f(p) + f(q) = f(p+q)$ for all $p, q \in S$, then $f(2) = 2$ and $f(3) = 3$.*

Proof. Suppose $f(2) \neq 2$. Let p_0 be a prime that is not the larger element of a twin prime pair. We see that

$$f(2)f(p_0) = f(2p_0) = f(p_0) + f(p_0) = 2f(p_0).$$

Hence, $(f(2) - 2)f(p_0) = 0$. However, $f(2) \neq 2$. Thus, $f(p_0) = 0$.

Suppose $5 \in S$. Then, $f(4) = f(2) + f(2) = 2f(2)$ and $f(5) = f(2) + f(3) = f(2)$. In addition,

$$2f(2)^2 = f(4)f(5) = f(20) = f(3) + f(17) = 0 + 0 = 0.$$

Hence, $f(2) = 0$. If p_0 is the larger element of a twin prime pair, then

$$f(p_0) = f(2) + f(p_0 - 2) = 0,$$

implying that f vanishes on all primes.

Suppose $7 \in S$. Then,

$$f(2)f(7) = f(14) = f(3) + f(11) = 0 + 0 = 0.$$

At least one of $f(2), f(7)$ is zero. Once again, $f(4) = 2f(2)$ and $f(5) = f(2)$. Note that

$$f(20) = f(7) + f(13) = f(7)$$

and

$$f(20) = f(4)f(5) = 2f(2)^2.$$

So, $f(2) = 0$ if and only if $f(7) = 0$. Hence, $f(2) = f(7) = 0$. This implies that f vanishes on the primes, which is impossible. Thus, $f(2) = 2$.

We split the proof of $f(3) = 3$ into two cases, depending on whether $5 \in S$ or $7 \in S$. In both parts, we write $f(n)$ in terms of $f(3)$ for multiple values of n, then use these values to solve for $f(3)$. In addition, we note that $f(4) = 2f(2) = 4$ and $f(5) = f(3) + 2$.

(1) Suppose $5 \in S$, we have

$$f(7) = f(2) + f(5) = f(3) + 4,$$

$$f(14) = f(2)f(7) = 2f(3) + 8,$$

$$f(11) = f(14) - f(3) = f(3) + 8,$$

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$$\begin{aligned} f(20) &= f(4)f(5) = 4f(3) + 8, \\ f(17) &= f(20) - f(3) = 3f(3) + 8, \\ f(22) &= f(2)f(11) = 2f(3) + 16, \\ f(22) &= f(5) + f(17) = 4f(3) + 10, \\ 4f(3) + 10 &= 2f(3) + 16, \\ f(3) &= 3. \end{aligned}$$

(2) Suppose $7 \in S$, we have

$$\begin{aligned} f(10) &= f(2)f(5) = 2f(3) + 4, \\ f(7) &= f(10) - f(3) = f(3) + 4, \\ f(14) &= f(2)f(7) = 2f(3) + 8, \\ f(11) &= f(14) - f(3) = f(3) + 8, \\ f(13) &= f(2) + f(11) = f(3) + 10, \\ f(26) &= f(2)f(13) = 2f(3) + 20, \\ f(23) &= f(26) - f(3) = f(3) + 20, \\ f(20) &= f(4)f(5) = 4f(3) + 8, \\ f(17) &= f(20) - f(3) = 3f(3) + 8, \\ f(34) &= f(2)f(17) = 6f(3) + 16, \\ f(34) &= f(11) + f(23) = 2f(3) + 28, \\ 6f(3) + 16 &= 2f(3) + 28, \\ f(3) &= 3. \end{aligned}$$

□

We extend this lemma a little further.

Lemma 2.4. *Using the same conditions as before, $f(n) = n$ for all $n \leq 23$.*

Proof. We proceed by induction starting from the fact that $f(n) = n$ for all $n < 5$. If n is not a prime power, then $n = ab$, with $\gcd(a, b) = 1$ and $a, b > 1$. So, $f(n) = f(a)f(b) = n$. We only need to check that $f(n) = n$ when n is a prime power. The only possibilities are $n = 5, 7, 8, 9, 11, 13, 16, 17, 19, 23$. In both cases of the previous proof, $f(5) = f(3) + 2$, $f(7) = f(3) + 4$, $f(11) = f(3) + 8$, and $f(17) = 3f(3) + 8$. Therefore, $f(n) = n$ for all $n \in \{5, 7, 11, 17\}$. Here are the other

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primes:

$$f(13) = f(11) + f(2) = 13,$$
$$f(19) = f(17) + f(2) = 19,$$
$$f(23) = f(26) + f(3) = f(2)f(13) + 2 = 23.$$

The only remaining cases are $n = 8, 9, 16$. If $5 \in S$, then

$$f(8) = f(5) + f(3) = 8,$$
$$f(16) = f(11) + f(5) = 16,$$
$$f(9) = \frac{f(72)}{f(8)} = \frac{f(67) + f(5)}{8} = \frac{(f(70) - f(3)) + 5}{8} = \frac{f(2)f(5)f(7) + 2}{8} = 9.$$

If $7 \in S$, then

$$f(8) = f(24)/f(3) = (f(17) + f(7))/3 = 8,$$
$$f(9) = f(7) + f(2) = 9,$$
$$f(16) = \frac{f(48)}{f(3)} = \frac{f(37) + f(11)}{3} = \frac{(f(40) - f(3)) + 11}{3}$$
$$= \frac{f(5)f(8) + 8}{3} = 16. \qquad \square$$

In addition to proving that the primes are AU unconditionally, Spiro also found a shorter proof conditional on Goldbach's Conjecture. Similarly, we may prove Theorem 1.2 using a variant of Goldbach's Conjecture, which we write here.

Conjecture 2.5. *If n is a composite prime power other than* 4, 8, 9, 49, *and* 64, *then $2n$ is the sum of two primes that are not the larger elements of a twin prime pair.*

Theorem 2.6. *Assuming the conclusion of Conjecture* 2.5 *for all numbers $2n$ with $n \leq M$, $f(n) = n$ for all $n \leq M$.*

Proof. We already know that $f(n) = n$ for all $n \leq 23$. We proceed by induction. Suppose $f(n) = n$ for all $n < m$ with $23 < m \leq M$. We show that $f(m) = m$. If m is not a prime power, then $m = ab$ for some relatively prime a, b with $a, b > 1$. Therefore,

$$f(m) = f(a)f(b) = ab = m.$$

If m is the larger element of a twin prime pair, then

$$f(m) = f(2) + f(m - 2) = 2 + (m - 2) = m.$$

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Suppose m is a prime for which $m-2$ is not prime. Then, $m+q \equiv 2 \bmod 4$ for some $q \in \{3, 17\}$. In this case, $(m+q)/2$ is an odd number. Because $(m+q)/2 < m$, we have

$$f(m+q) = f(2)f\left(\frac{m+q}{2}\right) = 2\left(\frac{m+q}{2}\right) = m+q,$$

which implies that

$$f(m) = f(m+q) - f(q) = m.$$

Suppose m is a prime power other than 49 and 64. If m is a power of 2, then $m = p+q$ for some prime p, q with $p-2, q-2$ not prime. By our inductive assumption, $f(m) = m$. Suppose m is an odd prime power. Then, $2m = p+q$ with $p-2, q-2$ not prime. Let $p < q$. Then, $p < m < q < 2m$. We know that $f(p) = p$, so it suffices to show that $f(q) = q$. There exists an $r \in \{3, 17, 23, 29\}$ such that $q + r \equiv 4 \bmod 8$. Thus,

$$f(q) + f(r) = f(q+r) = f(4)f\left(\frac{q+r}{4}\right) = q+r.$$

We obtain $f(q) = q$ and $f(m) = m$.

Finally, we consider $n = 49$ and $n = 64$

$$\begin{aligned} f(49) &= \frac{f(196)}{f(4)} = \frac{f(179)+f(17)}{4} = \frac{(f(182)-f(3))+17}{4} \\ &= \frac{f(2)f(7)f(13)+14}{4} = 49, \\ f(128) &= \frac{f(384)}{f(3)} = \frac{f(355)+f(29)}{3} = \frac{f(5)f(71)+29}{3} \\ &= \frac{(5\cdot 71)+29}{3} = 128. \end{aligned}$$

□

3. An Unconditional Proof

Computational tests show that Conjecture 2.5 holds for all prime powers less than 10^{16}.[a] Therefore, $f(n) = n$ for all $n \le 10^{16}$. Our goal for the rest of the paper is to show that $f(n) = n$ for all n. First, we show that $f(n) = n$ on a specific set H containing the primes. Then, we show that if n is the smallest number satisfying $f(n) \neq n$, then there exists a number m with $\gcd(m, n) = 1$ and $mn = p+q$ with p, q prime and $p-2, q-2$ prime. This will imply that $f(n) = n$ by induction.

Theorem 3.1. *Let*

$$H = \{n : v_p(n) \le 1 \text{ if } p > 1000; p^{v_p(n)+1} < 10^9 \text{ if } p < 1000\}.$$

Then, $f(n) = n$ for all $n \in H$.

[a]Contact the author for PARI/GP code.

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The following lemma is a variant of Spiro's Lemma 5. Throughout the proof of the lemma, we use $\pi_2(x)$ to refer to the number of twin prime pairs with smaller element at most x.

Lemma 3.2. *For every prime $p > 10^{16}$, there is an odd prime $q < p$ with $q-2$ not prime such that $p+q \in H$.*

Proof. Let $p > 10^{16}$ be prime and $N(p)$ be the number of primes $q < p$ such that $p+q \in H$. In the proof of [7, Lemma 10], Spiro showed that if $p > 10^{10}$, then

$$N(p) \geq 0.3\frac{p-1}{\log(p-1)}.$$

Suppose $p > 10^{16}$. By [5, Theorem 5.14],

$$\pi_2(p) \leq 7.68\frac{p}{\log^2 p}.$$

Because $p > 10^{16}$, $\log p > 36.8$, implying that

$$\pi_2(p) \leq 0.21\frac{p}{\log p}.$$

For $p > 10^{16}$,

$$\frac{p-1}{\log(p-1)}\left(\frac{p}{\log p}\right)^{-1} = \left(1-\frac{1}{p}\right)\frac{\log p}{\log(p-1)} > 1-\frac{1}{p} > 1-10^{-16}.$$

Therefore,

$$N(p) \geq 0.3\frac{p-1}{\log(p-1)} > 0.3(1-10^{-16})\frac{p}{\log p} > 0.29\frac{p}{\log p},$$

which gives us

$$N(p) - \pi_2(p) > 0.08\frac{p}{\log p} \geq 0.08\frac{10^{16}}{\log(10^{16})} = 2.17 \cdot 10^{13}.$$

Because this difference is greater than 1, there exists an odd prime $q < p$ with $q-2$ not prime such that $p+q \in H$. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1. We proceed by induction on H. Let $m \in H$ with $m > 10^{16}$. Assume that $f(n) = n$ for all $n \in H$ with $n < m$. Assume that m is not a prime power. Then, $f(m) = f(a)f(b)$ with $\gcd(a,b) = 1$ and $a, b > 1$. Note that $a, b \in H$ because the exponents of their prime factors are at most as large as they are in m.

Suppose m is a prime power. Then, $m = p^\alpha$ for some prime p. If $p < 1000$, then $p^\alpha < 10^9$, which is impossible. Thus, $p > 1000$. In this case, $\alpha = 1$. Hence, m is prime. If $m-2$ is prime, then $f(m-2) = m-2$ because $m-2 \in H$. Therefore, $f(m) = f(m-2) + f(2) = m$.

Suppose $m-2$ is composite. There exists an odd prime $q < m$ with $q-2$ not prime such that $m+q \in H$. Clearly, $m+q$ is even. However, $m+q$ is not a power of 2 because every power of 2 in H is less than 10^9. We may write $m+q = 2^k r$ with r odd and $r \in H$. Because $r < m$,

$$f(m) = f(m+q) - f(q) = f(2^k)f(r) - f(q) = 2^k r - q = (m+q) - q = m. \qquad \square$$

Now we know that $f(n) = n$ for all $n \in H$, we may show that $f(n) = n$ for all n.

Proof of Theorem 1.2. We prove the statement for all $n \notin H$. Suppose n is not a multiple of 3. Then, the set of even elements $r \in H$ for which $r \equiv 1 \bmod 3$ and $\gcd(r, n) = 1$ has positive density [6, §174] because H contains all square-free numbers. Almost all even numbers m can be expressed as a sum of two primes [4]. However, it is also almost always the case that $m-2$, $m-3$, and $m-5$ are composite. Therefore, almost all even numbers can be expressed as the sum of two primes greater than 5. We have

$$rn = p + q$$

with p, q primes greater than 5 for some r. Because $rn \equiv 1 \bmod 3$, we have $p, q \equiv 2 \bmod 3$. This implies that p and q are not the larger elements of twin prime pairs. Hence,

$$f(r)f(n) = f(rn) = f(p+q) = f(p) + f(q).$$

Because $p, q, r \in H$, $f(p) = p$, $f(q) = q$, and $f(r) = r$. Therefore, $f(n) = n$.

Since f is multiplicative, we only need to show that $f(n) = n$ for all powers of 3 to finish the proof. Let $n = 3^\alpha$. Let p be a prime congruent to 1 mod 3 that is not the larger element of a twin prime pair (such as 37). By Dirichlet's Theorem, there exists a prime q satisfying $3^\alpha \parallel p+q$. Then, $p+q = 3^\alpha r$ with $3 \nmid r$. In addition, p and q are not the larger elements of twin prime pairs. Hence, $f(3^\alpha) = 3^\alpha$. Therefore, f is the identity function. $\square$

4. Conclusion

The study of additive uniqueness still has many avenues for future research. While there have been recent papers investigating the additive uniqueness of certain sets (such as [1–3, 7]), there are few general theorems. For example, can an AU set be arbitrarily sparse? Spiro answered this question in the affirmative when we only require f to be nonvanishing on the integers, but not necessarily the primes [7, Theorem 2]. But, the more general question remains open.

If $1 \notin S$, then we can obtain a lower bound for the counting function of S.

Lemma 4.1. *Let S be an AU set that does not contain 1. Then, for every prime p, either $p \in S$ or there exist $x, y \in S$ such that $x + y = p$.*

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Proof. Suppose there exists a prime p such that $p \notin S$ and there do not exist $x, y \in S$ such that $x + y = p$. Then, the function

$$f(n) = \begin{cases} 1, & \text{if } n \in \{1, p\}, \\ 0, & \text{otherwise} \end{cases}$$

is a nonvanishing non-identity multiplicative function with the property that $f(x) + f(y) = f(x + y)$ for all $x, y \in S$. □

Corollary 4.2. *For an AU set S not containing* 1,

$$\#(S \cap [1, x]) \gg \sqrt{\frac{x}{\log x}}.$$

Proof. By the previous lemma, $(S \cup \{0\}) + (S \cup \{0\})$ contains the primes. Therefore,

$$\pi(x) \leq \#((S \cup \{0\}) + (S \cup \{0\})) \leq (\#(S \cup \{0\}))^2.$$

Because $\pi(x) \sim x/\log x$, we have $S \gg \sqrt{x/\log x}$. □

Is this bound optimal? In addition, how sparse can an AU set containing 1 be? Corollary 4.2 does not hold when $1 \in S$. Chung and Phong [3] have shown that the tetrahedral numbers are AU (and have conjectured that the k-tetrahedral numbers are AU for $k > 3$). As far as the author is aware, the tetrahedral numbers are the sparsest AU set currently known.

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