On ordered factorizations into distinct parts II

Noah Lebowitz-Lockard
University of Texas, Tyler, TX 75799
nlebowi@gmail.com
May 8, 2022

Abstract

Let \( G(n) \) be the number of ordered factorizations of \( n \) into distinct parts greater than 1. We improve the accuracy of the Dirichlet series of \( G \), then use this result to obtain a more precise asymptotic formula for the sum of \( G(n) \) for all \( n \leq x \).

1 Introduction

Let \( g(n) \) be the number of ordered factorizations of \( n \) into parts greater than 1. In other words, \( g(n) \) is the number of ordered tuples of integers \((a_1, \ldots, a_k)\) satisfying \( a_1 \cdots a_k = n \) and \( a_i > 1 \) for all \( i \). For example, \( g(16) = 7 \) because the factorizations of 16 are

\[
16, \quad 8 \cdot 2, \quad 2 \cdot 8, \quad 4 \cdot 4, \quad 4 \cdot 2 \cdot 2, \quad 2 \cdot 4 \cdot 2, \quad 2 \cdot 2 \cdot 4.
\]

In addition, we let \( G(n) \) be the number of such factorizations into distinct parts. So, \( G(16) = 3 \) because the only acceptable factorizations are

\[
16, \quad 8 \cdot 2, \quad 2 \cdot 8.
\]

Kalmár [Kal31] first investigated the statistical properties of \( g(n) \) in the 1930’s, showing that

\[
\sum_{n \leq x} g(n) \sim -\frac{1}{\rho \zeta'(\rho)} x^\rho,
\]

where \( \zeta \) is the Riemann zeta function and \( s = \rho \approx 1.73 \) is the unique real number greater than 1 satisfying \( \zeta(s) = 2 \). Ikehara [Ike41] subsequently reduced the error term. Most recently, Hwang [Hwa00] proved that

\[
\sum_{n \leq x} g(n) = -\frac{1}{\rho \zeta'(\rho)} x^\rho + O(x^\rho \exp(-c(\log \log x)^{(3/2)-\epsilon})),
\]

for all \( \epsilon > 0 \), where \( c = c(\epsilon) \) is a positive constant.
Research on \( G(n) \) began much more recently. In 1993, Warlimont found the following upper bound for the sum of \( G(n) \):
\[
\sum_{n \leq x} G(n) \leq x \cdot L(x)^{O(1)},
\]
where
\[
L(x) = \exp \left( \frac{\log x \log_3 x}{\log_2 x} \right).
\]
(From here on, \( \log_k \) refers to the \( k \)th iterate of the logarithm.) Unfortunately, he did not find a lower bound of the same shape. The author and Pollack [LP20] refined Warlimont’s sum, obtaining
\[
\sum_{n \leq x} G(n) = x \cdot L(x)^{1+o(1)}.
\]
In this article, we refine this bound even further. In order to state this result, we use the Lambert \( W \) function, which is defined as the inverse of the function \( f(z) = ze^z \). Finally, we let \( G_{<N}(n) \) be the number of ordered factorizations of \( n \) into distinct elements of \([2, N - 1]\).

**Theorem 1.1.** Fix \( \epsilon > 0 \). The sums
\[
\sum_{n \leq x} G_{\log x/\log_2 x}(n), \quad \sum_{n \leq x} G(n)
\]
both lie between
\[
x \exp \left( \frac{\log x \log_3 x}{\log_2 x} - \frac{\log x \log_4 x}{\log_2 x} - (1 + o(1)) \frac{\log x}{\log_2 x} \right)
\]
and
\[
x \exp \left( \frac{(\log x)W(\log_2 x)}{\log_2 x} + (2 + o(1)) \frac{(\log x)(\log_3 x)^2}{(\log_2 x)^2} \right).
\]

For a given \( z \), the asymptotic expansion of \( W(z) \) is
\[
\log z - \log_2 z + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^k}{m!} \binom{k + m}{k + 1} \frac{(\log_2 z)^m}{(\log z)^{k+m}},
\]
where
\[
\binom{k + m}{k + 1}
\]
refers to a Stirling number of the first kind [CGH+96, p.349]. The first few terms of our upper bound are
\[
x \exp \left( \frac{\log x}{\log_2 x} \left( \log_3 x - \log_4 x + \frac{\log_4 x}{\log_3 x} + \frac{(\log_4 x)^2}{(\log_3 x)^2} + \cdots \right) \right).
\]
Note that the first two terms in the upper and lower bounds match. (For the corresponding sums for unordered factorizations, see [Opp27, ST33, War93].)

The lower bound comes from elementary methods. However, the upper bound comes from a new asymptotic formula for the Dirichlet series of $G$, which may be of independent interest. This improves on similar results [War93, LP20]. From here on, we let $G(s)$ and $G_{<N}(s)$ be the Dirichlet series of $G$ and $G_{<N}$, respectively.

**Theorem 1.2.** Let $S = 1/(1-s)$. Let $s \to 1^+$ and $N$ grow more quickly than any constant multiple of $S^{3S}$. Then,

$$\exp(\exp(S \log S - (1 + o(1)) \log S)) \leq G_N(s) \leq G(s) \leq \exp(\exp(S \log S + (1 + o(1)) \log S)).$$

We can also write this result more compactly as

$$e^{(S^{S-1+o(1)})} \leq \sum_{n=1}^{\infty} \frac{G_N(n)}{n^s} \leq \sum_{n=1}^{\infty} \frac{G(n)}{n^s} \leq e^{(S^{S+1+o(1)})}.$$

## 2 The Dirichlet series

Let $G(s)$ be the Dirichlet series of $G$. Warlimont [War93, Eq. (7)] showed that

$$G(s) = \int_0^\infty e^{-x} \left( \prod_{m=2}^{\infty} \left(1 + \frac{x}{m^s}\right) \right) \, dx.$$  

Using this result, he obtained upper and lower bounds for reasonably small $s$. (From here on, we let $S = 1/(1-s).$)

**Theorem 2.1.** For all $s \in (1, 2]$ and $x \geq 30$, we have

$$\exp(\exp((1/2)S \log S)) \leq G(s) \leq \exp(\exp(2S \log S))$$

for all $s \in (1, 2]$ and all $x \geq 30$.

We, however, are only concerned with the asymptotic behavior of this sum as $s \to 1^+$, rather than explicit bounds. From [LP20, Section 2.2], we have

$$G(s) \leq \exp(\exp((1 + o(1))S \log S)).$$

Over the next two sections, we improve this bound and establish a similar lower bound. Our arguments are refinements of those in [War93, Section 2].
3 A lower bound for the Dirichlet series

In this section, we bound the Dirichlet series from above by refining the central argument of [War93, p.191]. In order to make the result stronger, we show that it still holds when we force the elements of our factorization to be reasonably small. For a given \( N \), let \( G_{<N}(n) \) be the number of ordered factorizations of \( n \) into distinct parts in \((1, N)\) and \( G_{<N} \) the corresponding Dirichlet series.

**Lemma 3.1.** Fix two large positive constants \( M \) and \( N \) with \( M < N \). As \( s \to 1^+ \), we have

\[
G_{<N}(s) \geq \exp \left( -1 + \frac{S}{M^{s-1}} - \frac{S}{N^{s-1}} - \frac{1}{2(2s-1)M^{2s-1}} \right).
\]

**Proof.** We have

\[
G(s) \geq \int_M^{\infty} e^{-x} \left( \prod_{m \geq M} \left( 1 + \frac{M}{m^s} \right) \right) dx.
\]

Because \( M \leq m < m^s \), we may bound the inner terms of the product using the Taylor series of \( \log \):

\[
1 + \frac{M}{m^s} \geq \exp \left( \frac{M}{m^s} - \frac{1}{2} \frac{M^2}{m^{2s}} \right).
\]

Therefore,

\[
G_N(s) \geq \int_M^{\infty} e^{-x} \left( \prod_{M \leq m < N} \left( 1 + \frac{M}{m^s} \right) \right) dx
\]

\[
= \exp \left( -M + \sum_{M \leq m < N} \log \left( 1 + \frac{M}{m^s} \right) \right)
\]

\[
\geq \exp \left( -M + \sum_{M \leq m < N} \frac{M}{m^s} - \frac{1}{2} \sum_{m \geq M} \frac{M^2}{m^{2s}} \right)
\]

\[
\geq \exp \left( M \left( -1 + \frac{S}{M^{s-1}} - \frac{S}{N^{s-1}} - \frac{1}{2(2s-1)M^{2s-1}} \right) \right). \quad \square
\]

**Theorem 3.2.** Let \( s \to 1^+ \) and let \( N \) grow more quickly than any constant multiple of \( S^{3S} \). Then,

\[
G_{<N}(s) \geq \exp(\exp(S \log S - (1 + o(1)) \log S)).
\]

**Proof.** We take the result from the previous lemma and optimize \( M \). We will show that \( N \) is large enough to ensure that its effect it has on the bound is negligible. For simplicity, we do not consider the last term here, as it will turn out to be negligible. Note that

\[
\frac{d}{dM} \left( M \left( -1 + \frac{1}{(s-1)M^{s-1}} \right) \right) = -1 + \frac{2 - s}{s - 1} M^{-(s-1)}.
\]
Setting this quantity equal to 0 gives us
\[ M = \left( \frac{2 - s}{s - 1} \right)^{1/(s-1)} = \exp \left( \frac{1}{s - 1} \log \frac{1}{s - 1} + \frac{1}{s - 1} \log(2 - s) \right). \]

Note that \( 2 - s = 1 - (s - 1) \). As \( s \to 1^+ \), we have
\[ \log(2 - s) = \log(1 - (s - 1)) \sim -(s - 1), \]
\[ M = \exp(S \log S - 1 + o(1)). \]

Plugging this value of \( M \) into our lower bound from the previous lemma shows that
\[ G_N(s) \geq \exp(S \log S - 1 + o(1)) \left( -1 + 1 \frac{(s - 1)^2}{2(2s - 1)(2 - s)^2} \right). \]

Note that
\[ -1 + \frac{1}{2 - s} = (s - 1) + \frac{(s - 1)^2}{2 - s} = (s - 1) + (1 + o(1))(s - 1)^2. \]

Because \( N \) is sufficiently large, \( S/N^{s-1} = o((s - 1)^2) \). Therefore,
\[ -1 + \frac{1}{2 - s} - \frac{S}{N^{s-1}} - \frac{(s - 1)^2}{2(2s - 1)(2 - s)^2} = (s - 1) + o((s - 1)^2) = \exp(-(1 + o(1)) \log S), \]

giving us
\[ G_{\leq N}(s) \geq \exp(S \log S - (1 + o(1)) \log S). \]

\[ \square \]

4 The upper bound

By modifying the argument in [War93, p.192], we obtain a similar upper bound.

**Theorem 4.1.** As \( s \to 1^+ \), we have
\[ G(s) \leq \exp(\exp(S \log S + (1 + o(1)) \log S)). \]

**Proof.** Warlimont showed that
\[ G(s) \leq \int_0^\infty (1 + x)^M \exp \left( x \left( -1 + \frac{1}{(s - 1)M^{s-1}} \right) \right) dx. \]

For this integral to be finite, we only need to have \( M^{s-1} > (s - 1)^{-1} \). We will soon optimize this. Let
\[ C = 1 - \frac{1}{(s - 1)M^{s-1}}. \]
Split the integral into two parts. First, we bound the integral over \([0, M]\):

\[
\int_0^M (1 + x)^M e^{-Cx} \, dx \leq M \max_{0 \leq x \leq M} (1 + x)^M e^{-Cx}.
\]

We maximize this function setting the derivative to 0:

\[
\frac{d}{dx} (1 + x)^M e^{-Cx} = (1 + x)^{M-1} e^{-Cx} (M - C(1 + x)).
\]

We obtain \(x = (M/C) - 1\), which gives us a bound of

\[
e^{CM} \left( \frac{M}{eC} \right)^M.
\]

Consider the interval \([M, \infty)\). In this case, \((1 + x)^M \leq e^{xM}\). We bound our integral as

\[
e \int_0^\infty x^M e^{-Cx} \, dx.
\]

By changing variables, we can rewrite this integral in terms of the Gamma function. Let \(y = Cx\). In this format, the integral is at most

\[
Ce \int_0^\infty \left( \frac{y}{C} \right)^M e^{-y} \, dy = \frac{e \Gamma(M+1)}{C^{M-1}} \sim eC^2 \sqrt{2\pi} \left( \frac{M+1}{eC} \right)^{M+1}.
\]

Our goal at this point is to minimize \(((M+1)/(eC))^{M+1}\). For simplicity, we select \(M\) so that \((M/(eC))^{M}\) is as small as possible. Ideally, \(M\) should be small, but it still must obey the restriction that \(M^{s-1} > 1/(s-1)\). If \(M\) gets too close to this quantity from above, then \(1/C\) goes to \(\infty\).

Suppose

\[
M = \exp(S \log S + h(S))
\]

with \(h(S) = o(S \log S)\). From here, we optimize \(h(S)\). In this case,

\[
\left( \frac{M}{eC} \right)^M = \exp(M(\log M - (1 + \log C)))
\]

\[
= \exp(\exp(S \log S + h(S))(S \log S - 1 + h(S) - \log C))
\]

\[
= \exp(\exp(S \log S + h(S) + \log(S \log S + h(S) - (\log C) - 1))).
\]

To finish the proof, we need to choose \(h(S)\) to minimize

\[
h(S) + \log(S \log S + h(S) - (\log C) - 1).
\]

In addition, we have

\[
\log C = \log \left( 1 - \frac{1}{(s-1)M^{s-1}} \right) \sim - \frac{1}{(s-1)M^{s-1}} = - \frac{1}{\exp(S^{-1}h(S))}.
\]
Suppose \( h(S) = o(\log S) \), but still goes to \( \infty \) with \( S \). Then,

\[
h(S) + \log(S \log S + h(S) - (\log C) - 1) \sim \log S.
\]

Putting everything together gives us

\[
(M/(eC))^M = \exp(\exp(S \log S + (1 + o(1)) \log S)).
\]

Replacing \( M^M \) with \((M + 1)^{M+1}\) does not change this bound.

5 The original sum

Now that we have our bounds on the Dirichlet series, we can return our focus to the sum of \( G(n) \). When the author and Pollack originally bounded this sum [LP20], the lower bound came from an elementary argument, whereas we wrote both an elementary and analytic argument for the upper bound. For the upper bound, we refine the argument from [LP20, p.3] using Theorem 4.1. For the lower bound, we refine [LP20, p.4].

Recall that \( W \) refers to the Lambert \( W \) function, the inverse of the function \( f(z) = ze^z \).

**Theorem 5.1.** As \( s \to 1^+ \), we have

\[
\sum_{n \leq x} G(n) \leq x \exp \left( \frac{(\log x) W(\log_2 x)}{\log_2 x} + (2 + o(1)) \frac{(\log x)(\log_3 x)^2}{(\log_2 x)^2} \right).
\]

**Proof.** We have

\[
\sum_{n \leq x} G(n) \leq x^s \sum_{n \leq x} \frac{G(n)}{n^s} \leq x^s G(s) \leq x \exp(S^{-1} \log x + \exp(S \log S + (1 + o(1)) \log S)).
\]

We choose \( s \) in order to minimize this bound. In [LP20], we let \( s = 1 + (1 + \epsilon) \log_3 / \log_2 x \) so that \( S = (1 + \epsilon)^{-1} \log_2 x / \log_3 x \). Here, we modify this choice. Assume \( S \sim \log_2 x / \log_3 x \). If \( \exp(S \log S + \log S) \approx S^{-1} \log x \), then \( S \approx \exp(W(\log_2 x \log_2 x)) \). Rather than use this value, we set

\[
S = \exp(W(\log_2 x - (2 + \epsilon) \log_3 x)) = (\log_2 x - (2 + \epsilon) \log_3 x)/W(\log_2 x - (2 + \epsilon) \log_3 x).
\]

because we have \((1 + o(1)) \log S\) instead of \( \log S \) in the rightmost exponential.

We now have

\[
S^{-1} \log x = \frac{(\log x) W(\log_2 x)}{\log_2 x} + (2 + \epsilon + o(1)) \frac{(\log x)(\log_3 x)^2}{\log_2 x}.
\]

In addition,

\[
S \log S = \log_2 x - (2 + \epsilon) \log_3 x,
\]

\[
\log S = (1 + o(1)) \log_3 x.
\]
This gives us

\[ \exp(S \log S + (1 + o(1)) \log S) = \frac{\log x}{(\log_2 x)^{1+o(1)}}, \]

which is negligible. Putting everything together and letting \( \epsilon \to 0 \) produces our upper bound.

**Theorem 5.2.** We have

\[ \sum_{n \leq x} G_{< y/\log_2 x}(n) \geq x \exp \left( \frac{\log x \log_3 x}{\log_2 x} - \frac{\log x \log_4 x}{\log_2 x} - (1 + o(1)) \frac{\log x}{\log_2 x} \right). \]

**Proof.** Fix a number \( y \) with \( y^y > x \) and let \( k = \lfloor y \rfloor \). From [LP20, p.4], we have

\[ \sum_{n \leq x} G_{< y}(n) \geq x (\log(x^{1/y}) - \log k - 1)^k. \]

The author and Pollack showed that

\[ \sum_{n \leq x} G(n) \geq x \exp \left( (1 + o(1)) \frac{\log x \log_3 x}{\log_2 x} \right), \]

by setting

\[ y = (1 - \epsilon) \frac{\log x}{\log_2 x}. \]

Once again, we obtain a better bound by optimizing \( y \).

We choose \( y \) to maximize

\[ (\log(x^{1/y}) - \log y - 1)^y = \exp(y \log(\log(x^{1/y}) - \log y - 1)). \]

Note that

\[ \frac{d}{dy} \left( y \log \left( \frac{\log x}{y} - \log y - 1 \right) \right) \]

\[ = \log \left( \frac{\log x}{y} - \log y - 1 \right) - \left( \frac{\log x}{y} - \log y - 1 \right)^{-1} \left( \frac{\log x}{y} + 1 \right). \]

Setting this quantity equal to 0 gives us

\[ \frac{\log x}{y} - \log y - 1 = \exp \left( W \left( \frac{\log x}{y} + 1 \right) \right). \]

As this equation is difficult to solve for \( y \), we note that \( y \sim \log x / \log_2 x \), which implies that \( (\log x)/y \sim \log_2 x \). In light of this fact, we solve

\[ \frac{\log x}{y} - \log y - 1 = \exp(W(\log_2 x)) \]
instead. Another advantage of this choice is that the exponential has a $W(\log_2 x)$ term, just as it did in the upper bound. The solution is

$$y = \frac{\log x}{W((\log x)e^{W(\log_2 x)+1})}.$$  

Putting everything together gives us

$$\sum_{n \leq x} G_{< \log x}(n) \geq x \exp(W(\log_2 x)y),$$

with $y$ defined above. In order to finish the proof, we need to write the first few terms of the asymptotic expansion of $y$. To start, we note that

$$e^{W(\log_2 x)} = (\log_2 x)/W(\log_2 x),$$

giving us

$$y = \frac{\log x}{W((\log x)e^{((\log_2 x)/W(\log_2 x))+1})}.$$

In addition,

$$(\log x) \exp \left( \frac{\log_2 x}{W(\log_2 x)} \right) = (\log x) \exp \left( (1 + o(1)) \frac{\log_2 x}{\log_3 x} \right) = (\log x)^{1+(1+o(1))/\log_3 x}.$$  

Plugging this into $W$ gives us

$$W((\log x)^{1+(1+o(1))/\log_3 x}) = \left( 1 + (1 + o(1)) \frac{1}{\log_3 x} \right) (\log_2 x) - (1 + o(1)) \log_3 x$$

$$= \log_2 x + (1 + o(1)) \frac{\log_2 x}{\log_3 x},$$

which implies that

$$y = (\log x) \left( \log_2 x + (1 + o(1)) \frac{\log_2 x}{\log_3 x} \right)^{-1} = \frac{\log x}{\log_2 x} - (1 + o(1)) \frac{\log x}{\log_2 x \log_3 x}.$$  

Hence,

$$\sum_{n \leq x} G_{< \log x}(n) \geq x e^{W(\log_2 x)y}$$

$$= x \exp \left( \left( \frac{\log x}{\log_2 x} - (1 + o(1)) \frac{\log x}{\log_2 x \log_3 x} \right) (\log_3 x - \log_4 x + o(1)) \right)$$

$$= x \exp \left( \frac{\log x \log_3 x}{\log_2 x} - \frac{\log x \log_4 x}{\log_2 x} - (1 + o(1)) \frac{\log x}{\log_2 x} \right). \quad \square$$
References


